A UNIVERSALITY THEOREM FOR RATIOS OF RANDOM CHARACTERISTIC POLYNOMIALS

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Abstract. We consider asymptotics of ratios of random characteristic polynomials associated with orthogonal polynomial ensembles. Under some natural conditions on the measure in the definition of the orthogonal polynomial ensemble we establish a universality limit for these ratios.

1. Introduction

1.1. **Formulation of the problem.** In this article we consider orthogonal polynomial ensembles of n particles on $\mathbb{R}$. Such ensembles are described by a positive Borel measure μ with finite moments, and the associated distribution function for the particles $\{x_1, \ldots, x_n\}$ of the form

$$d\mathbb{P}_{\mu,n}(x) = \frac{1}{Z_n} \triangle_n(x)^2 \prod_{i=1}^{n} d\mu(x_i).$$

Here Z_n is the normalization constant,

$$Z_n = \int \ldots \int \triangle_n(x)^2 \prod_{i=1}^{n} d\mu(x_i),$$

and

$$\triangle_n(x) = \prod_{n \geq i > j \geq 1} (x_i - x_j)$$

is the Vandermonde determinant. For symmetric functions $f(x) = f(x_1, \ldots, x_n)$ of the $x_i's$,

$$\langle f(x) \rangle_\mu \equiv \frac{1}{Z_n} \int \ldots \int f(x) \triangle_n(x)^2 \prod_{i=1}^{n} d\mu(x_i)$$

denotes the average of f with respect to $d\mathbb{P}_{\mu,n}(x)$.

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Given a set of n points, $\{x_1, \dots, x_n\}$, let

$$D_n^{\{x_1,\dots,x_n\}}(\alpha) = \prod_{i=1}^{n} (\alpha - x_i),$$

considered as a polynomial in $\alpha \in \mathbb{C}$. When the x_i's are chosen to be the *random* particle locations for an orthogonal polynomial ensemble, this becomes a random polynomial which we denote simply by $D_n(\alpha)$. By extension from the random matrix case (see below), $D_n(\alpha)$ is known as the *characteristic polynomial* for the orthogonal polynomial ensemble.

The goal of the present paper is to study the large n asymptotics of the averages

$$\left\langle \frac{D_n(\alpha_1)\dots D_n(\alpha_k)}{D_n(\beta_1)\dots D_n(\beta_k)} \right\rangle_\mu . \tag{1.1}$$

In particular, we focus on the bulk of the support for measures μ, and aim to establish universality of the scaling limit of averages (1.1) under mild assumptions on μ. Our main result says roughly that for μ locally absolutely continuous with a bounded Radon-Nikodym derivative, universality for the reproducing kernel implies universality for (1.1). It should be emphasized that aside for the existence of moments, we make no global assumptions on μ. In particular, we do not assume that μ is globally absolutely continuous or that $\mathrm{supp}(\mu)$ is compact.

1.2. **Motivation and remarks on related works.** In the case when $d\mu(x) = e^{-V(x)}dx$ one can interpret the particles in the definition of the orthogonal polynomial ensembles as eigenvalues of a random Hermitian matrix taken from a Unitary Ensemble of Random Matrix Theory (RMT). The most well studied case is that of $V(x) = x^2$, called the Gaussian Unitary Ensemble. In this context random characteristic polynomials are indeed the characteristic polynomials of a random Hermitian matrix.

Averages of characteristic polynomials of random matrices are basic objects of interest in RMT, and were first considered by Andreev and Simons [1], Brezin and Hikami [6] and Keating and Snaith [13]. In particular, such averages are used to make predictions about zeros of the Riemann-zeta function on the critical line (see, for example, Keating and Snaith [13], Conrey, Farmer, Keating, Rubinstein and Snaith [8], the survey article by Keating and Snaith [14], and references therein). Averages (1.1) are related to certain important distribution functions studied in physics of quantum chaotic systems. Two examples involve the curvature distribution of energy levels of a chaotic system and the statistics of the local Green functions, in particular, the joint distribution of local denity of states; see Andreev and Simons [1] and references therein. Many other uses are described, for example, in Brezin [5].

For averages (1.1) a number of algebraic and asymptotic results is available in the literature. Papers by Baik, Deift, and Strahov [3], Fyodorov and Strahov [12] and Borodin and Strahov [7] give explicit determinantal representations for (1.1). These representations can be used for the asymptotic analysis as $n \to \infty$. In particular, the asymptotics of (1.1) were investigated in Strahov and Fyodorov [12] in the case when $d\mu(x) = e^{-V(x)}dx$, and $V(x)$ is an even polynomial. Strahov and Fyodorov [12] deal with the asymptotic in the bulk of the spectrum. Vanlessen [23] shows that for certain class of unitary ensembles of Hermitian matrices, averages (1.1) have universal asymptotic behavior at the origin of the spectrum. The asymptotic analysis in [12], and in [23] is based on the reformulation of an orthogonal polynomial problem as a Riemann-Hilbert problem by Fokas, Its and Kitaev [11]. The Riemann-Hilbert problem is then analyzed asymptotically using the noncommutative steepest-descent method introduced by Deift and Zhou, see Deift [9] and references therein.

In recent years, it has become evident that orthogonal polynomial ensembles play a role in probabilistic models other than RMT as well (see, for example, the survey paper by König [15]). As the relevant measures in these models are not necessarily of the form $e^{-V(x)}dx$, it is of interest to study the problem of universality limits for basic quantities of interest for a broader class of measures. Lubinsky's universality theorems regarding bulk universality for the reproducing kernel (see Lubinsky [16, Theorem 1.1], and [18, Theorem 1.1], and also Findley [10], Simon [19], Totik [22] and Avila-Last-Simon [2] for extensions of Lubinsky's results and methods) are important steps in this direction. Our goal in this paper is to establish the corresponding universality limits for averages (1.1) in the bulk of the support of μ.

1.3. **Description of the main result.** For $n = 0, 1, 2, \dots$ we introduce the orthonormal polynomials associated with μ,

$$p_n(x) = \gamma_n x^n + \dots, \quad \gamma_n > 0.$$

The orthonormality conditions are

$$\int p_j(x)p_k(x)d\mu(x) = \delta_{jk}.$$

The nth reproducing kernel (also known as the Christoffel-Darboux kernel) for μ is

$$K_n(x,y) = \sum_{k=0}^{n-1} p_k(x)p_k(y).$$

If $d\mu(t) = w(t)dt$ in a neighborhood of x, we also define the normalized kernel to be

$$\widetilde{K}_n(x,y) = w(x)K_n(x,y).$$

The Christoffel-Darboux formula enables one to rewrite $K_n(x, y)$ as

$$K_n(x,y) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y}. \tag{1.2}$$

The reproducing kernel plays a special role in the theory of orthogonal polynomial ensembles. This is because the correlation functions for the orthogonal polynomial ensemble can be expressed as determinants of a matrix whose entries are given by values of the reproducing kernel (see, for example, Deift [9]).

We say K_n has a universal limit at x, if for any $a, b \in \mathbb{C}$ we have

$$\lim_{n\to\infty} \frac{K_n(x + \frac{a}{\widetilde{K}_n(x,x)}, x + \frac{b}{\widetilde{K}_n(x,x)})}{K_n(x,x)} = \frac{\sin \pi(a-b)}{\pi(a-b)} = \mathbb{S}(a,b). \tag{1.3}$$

We say K_n has a *uniform universal limit* at x, if the limit in (1.3) is uniform for a, b in compact subsets of $\mathbb{C}$.

Two approaches introduced recently by Lubinsky ([16]-[18]) make it possible to establish universality for K_n under relatively mild conditions on μ. These approaches were further extended and generalized by Findley [10], Simon [19], Totik [22] and Avila-Last-Simon [2]. A typical result is:

Theorem 1.1. *Let μ be a probability measure on $\mathbb{R}$ with compact support that is regular in the sense of Stahl and Totik* [20]*. Suppose $x \in \mathrm{supp}(\mu)$ has a neighborhood, J, such that μ is absolutely continuous in J: $d\mu(t) = w(t)dt$ for $t \in J$. Assume further, that w is positive and continuous at x. Then uniformly for a, b in compact subsets of the complex plane, we have*

$$\lim_{n\to\infty} \frac{K_n(x + \frac{a}{\widetilde{K}_n(x,x)}, x + \frac{b}{\widetilde{K}_n(x,x)})}{K_n(x,x)} = \mathbb{S}(a,b). \tag{1.4}$$

In other words, K_n has a uniform universal limit at x.

Remarks.

(a) Theorem 1.1 implies universality in the bulk for the m-point correlation function for the orthogonal polynomial ensemble (see Lubinsky [17, Section 1]).

(b) The Theorem follows, for example, from [18, Theorem 1.1] combined with remark (e) there. The result was obtained previously, however, by both Simon [19] and Totik [22], using a modification of Lubinsky's approach from [16].

(c) The requirement of continuity can be relaxed to a Lebesgue point type condition, assuming boundedness and uniform positivity of w in a neighborhood of x. This is Theorem 1.2 of [18].

(d) Avila, Last and Simon [2], using a modification of the approach of [18], obtain universality for certain measures whose support is a positive Lebesgue measure Cantor set.

(e) While it seems that local absolute continuity of μ is almost sufficient for

universality at x, it is certainly not necessary: Breuer [4] has recently shown there are purely singular measures such that universality holds uniformly for x in an interval (and $a, b \in \mathbb{R}$). Of course, $\widetilde{K}_n(x,x)$ cannot be defined for purely singular measures, so the statement needs to be slightly modified (see [4] for details).

In this paper we prove an analogue of Theorem 1.1 for averages (1.1). Here is our main result.

Theorem 1.2. *Let μ be a probabilty measure on $\mathbb{R}$ with finite moments. Let $x \in \operatorname{supp}(\mu)$ be such that:*
(i) There exists an interval, J, with $x \in J$ such that μ is absolutely continuous in J: $d\mu(t) = w(t)dt$ for $t \in J$. Moreover, $w \in L^\infty(J, dt)$.
(ii) $w(x) > 0$ and x is a Lebesgue point of w, by which we mean that

$$\lim_{r \to 0+} \frac{1}{r} \int_{x-r}^{x+r} |w(t) - w(x)| dt = 0.$$

(iii) K_n has a uniform universal limit at x.

Under these assumptions, for any pairwise distinct α_i's and β_j's, such that $\alpha_1, \dots, \alpha_k \in \mathbb{R}$, $\beta_1, \dots, \beta_k \in \mathbb{C} \setminus \mathbb{R}$

$$\lim_{n \to \infty} \left\langle \prod_{j=1}^{k} \frac{D_n\left(x + \frac{\alpha_j}{\widetilde{K}_n(x,x)}\right)}{D_n\left(x + \frac{\beta_j}{\widetilde{K}_n(x,x)}\right)} \right\rangle_\mu = (-1)^{\frac{k(k+1)}{2}} \frac{\triangle(\beta,\alpha)}{\triangle(\beta)^2 \triangle(\alpha)^2} \det\left(\mathbb{W}(\beta_i, \alpha_j)\right)_{i,j=1}^{k},$$

where

$$\mathbb{W}(\beta,\alpha) = \frac{1}{\beta - \alpha} + \int_{-\infty}^{+\infty} \frac{\sin(\pi(s-\alpha))ds}{\pi(s-\alpha)(s-\beta)}.$$

As an immediate corollary of Theorems 1.1 and 1.2, we obtain

Corollary 1.3. *Let μ be a probability measure on $\mathbb{R}$ with compact support that is regular in the sense of Stahl and Totik. Suppose $x \in \operatorname{supp}(\mu)$ has a neighborhood, J, such that μ is absolutely continuous in J: $d\mu(t) = w(t)dt$ for $t \in J$, with w bounded in J. Assume further, that w is positive and continuous at x. Then for any pairwise distinct α_i's and β_j's, such that $\alpha_1, \dots, \alpha_k \in \mathbb{R}$, $\beta_1, \dots, \beta_k \in \mathbb{C} \setminus \mathbb{R}$*

$$\lim_{n \to \infty} \left\langle \prod_{j=1}^{k} \frac{D_n\left(x + \frac{\alpha_j}{\widetilde{K}_n(x,x)}\right)}{D_n\left(x + \frac{\beta_j}{\widetilde{K}_n(x,x)}\right)} \right\rangle_\mu = (-1)^{\frac{k(k+1)}{2}} \frac{\triangle(\beta,\alpha)}{\triangle(\beta)^2 \triangle(\alpha)^2} \det\left(\mathbb{W}(\beta_i, \alpha_j)\right)_{i,j=1}^{k},$$

Remarks.
(a) The function $\mathbb{W}(\beta,\alpha)$ can be rewritten as

$$\mathbb{W}(\beta,\alpha)=\begin{cases}\frac{e^{i\pi(\beta-\alpha)}}{\beta-\alpha}, & \operatorname{Im}\beta>0,\\ \frac{e^{-i\pi(\beta-\alpha)}}{\beta-\alpha}, & \operatorname{Im}\beta<0.\end{cases}$$

The formula above can be obtained from that of Theorem 1.2 by contour integration.
(b) If $d\mu(y)=\mathrm{const}\, e^{-V(y)}dy$, and $V(y)$ is an even polynomial, then Theorem 1.2 reduces to the result obtained in Strahov and Fyodorov [21, Section 2.4].
(c) Note that the assumptions of Theorem 1.2 do not include compact support of μ. The theorem essentially says that for any measure with finite moments, local absolute continuity combined with a uniform universal limit for K_n at x imply uniform limits for averages (1.1) at x.
(d) As explained in remark (e) after Theorem 1.1, there are purely singular measures for which K_n has universal limits. As the absolute continuity of μ around x is essential to the proof of Theorem 1.2 we, unfortunately, have nothing to say about this case.

The rest of the paper is devoted to the proof of Theorem 1.2. Namely, in Section 2 we present a useful algebraic formula for averages (1.1) (Theorem 2.1). This formula reduces the investigation of asymptotics of (1.1) to that of the Cauchy transform of the reproducing kernel,

$$\int\frac{K_n(t,\alpha)d\mu(t)}{t-\beta}.$$

Theorem 3.1 establishes universality for the Cauchy transform of the reproducing kernel. Theorem 1.2 (which is the main result of this work) is then a simple corollary of Theorem 2.1, and of Theorem 3.1.

2. A formula for ratios of characteristic polynomials

Theorem 2.1. *Let $1\leq k\leq n$, and assume that $\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k$ are pairwise distinct complex numbers. Moreover, assume that $\operatorname{Im}(\beta_j)\neq 0$ for $j=1,\dots,k$. Then we have*

$$\left\langle\prod_{j=1}^{k}\frac{D_n(\alpha_j)}{D_n(\beta_j)}\right\rangle_{\mu}=(-1)^{\frac{k(k+1)}{2}}\frac{\triangle(\beta,\alpha)}{\triangle(\beta)^2\triangle(\alpha)^2}\det\left(W_n(\beta_i,\alpha_j)\right)_{i,j=1}^{k},$$

where the two-point function $W_n(\beta,\alpha)$ is defined by

$$W_n(\beta,\alpha)=\frac{1}{\beta-\alpha}+\int\frac{K_n(t,\alpha)d\mu(t)}{t-\beta}.$$

Proof. A formula of two-point function type for the average of ratios of characteristic polynomials was obtained by Baik, Deift and Strahov [3, Section III].

Theorem 3.3 in [3] gives

$$\left\langle \prod_{j=1}^{k} \frac{D_n(\alpha_j)}{D_n(\beta_j)} \right\rangle_{\mu} = (-1)^{\frac{k(k-1)}{2}} \widetilde{\gamma}_{n-1}^{k} \frac{\triangle(\beta,\alpha)}{\triangle(\beta)^2 \triangle(\alpha)^2} \det\left(\widetilde{W}_n(\beta_i,\alpha_j)\right)_{i,j=1}^{k},$$

where

$$\widetilde{W}_n(\beta,\alpha) = \frac{\widetilde{h}_n(\beta)\pi_{n-1}(\alpha) - \widetilde{h}_{n-1}(\beta)\pi_n(\alpha)}{\beta-\alpha}.$$

In the formulae above $\pi_l(\alpha)$ is the lth monic orthogonal polynomial associated with μ, the function $\widetilde{h}_l(\beta)$ is the Cauchy transform of $\pi_l(\alpha)$,

$$\widetilde{h}_l(\beta) = \frac{1}{2\pi i} \int \frac{\pi_l(t) d\mu(t)}{t-\beta},$$

and

$$\widetilde{\gamma}_{n-1} = -2\pi i \gamma_{n-1}^2.$$

Clearly, the formula for the average of ratios of characteristic polynomials can be rewritten as

$$\left\langle \prod_{j=1}^{k} \frac{D_n(\alpha_j)}{D_n(\beta_j)} \right\rangle_{\mu} = (-1)^{\frac{k(k+1)}{2}} \frac{\triangle(\beta,\alpha)}{\triangle(\beta)^2 \triangle(\alpha)^2} \det\left(W_n(\beta_i,\alpha_j)\right)_{i,j=1}^{k},$$

where

$$W_n(\beta,\alpha) = \frac{\gamma_{n-1}}{\gamma_n} \frac{h_n(\beta)p_{n-1}(\alpha) - h_{n-1}(\beta)p_n(\alpha)}{\beta-\alpha},$$

and where

$$h_l(\beta) = \int \frac{p_l(t) d\mu(t)}{t-\beta}.$$

To obtain the formula for $W_n(\beta,\alpha)$ stated in the Theorem we note that

$$\begin{aligned}
&\frac{\gamma_{n-1}}{\gamma_n} \frac{h_n(\beta)p_{n-1}(\alpha) - h_{n-1}(\beta)p_n(\alpha)}{\beta-\alpha} \\
&= \frac{\gamma_{n-1}}{\gamma_n} \frac{1}{\beta-\alpha} \left(\left(\int \frac{p_n(t)d\mu(t)}{t-\beta} \right) p_{n-1}(\alpha) - \left(\int \frac{p_{n-1}(t)d\mu(t)}{t-\beta} \right) p_n(\alpha) \right) \\
&= \frac{1}{\beta-\alpha} \int \frac{t-\alpha}{t-\beta} K_n(t,\alpha) d\mu(t) \\
&= \frac{1}{\beta-\alpha} + \int \frac{K_n(t,\alpha) d\mu(t)}{t-\beta},
\end{aligned}$$

where in the second equality we have used formula (1.2), and in the third equality we have used the reproducing property of K_n. □

3. Universality for the Cauchy transform of the reproducing kernel

Theorem 3.1. *Let $x \in \operatorname{supp}(\mu)$ be such that conditions $(i)-(iii)$ of Theorem 1.2 are satisfied. Then for any $\alpha \in \mathbb{R}$, $\beta \in \mathbb{C} \setminus \mathbb{R}$*

$$\lim_{n\to\infty} \frac{1}{\widetilde{K}_n(x,x)} \int \frac{K_n\left(x+\frac{\alpha}{\widetilde{K}_n(x,x)}, t\right)}{t-x-\frac{\beta}{\widetilde{K}_n(x,x)}} d\mu(t) = \int\limits_{-\infty}^{+\infty} \frac{\mathbb{S}(\alpha,s)}{s-\beta} ds,$$

where

$$\mathbb{S}(\alpha,s) = \frac{\sin \pi(\alpha-s)}{\pi(\alpha-s)}.$$

Remark. In case of the Chebyshev weight for example,

$$d\mu(y) = w(y)dy, \;\; w(y) = \frac{1}{\sqrt{1-y^2}}, \;\; y \in (-1,1),$$

Theorem 3.1 can be checked by direct computations. These computations are similar to those of [17, Section 2], for the reproducing kernel associated with the Chebyshev weight.

Theorem 3.1 follows from the following

Lemma 3.2. *Fix $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C} \setminus \mathbb{R}$. Under the conditions of Theorem 1.2, for any $M \geq 2|\beta|$,*

$$\begin{aligned} &\limsup_{n\to\infty} \left| \frac{1}{\widetilde{K}_n(x,x)} \int \frac{K_n\left(x+\frac{\alpha}{\widetilde{K}_n(x,x)}, t\right)}{t-x-\frac{\beta}{\widetilde{K}_n(x,x)}} d\mu(t) - \int \frac{\mathbb{S}(\alpha,s)}{s-\beta} ds \right| \\ &\leq \left| \int_{\mathbb{R}\setminus[-M,M]} \frac{\mathbb{S}(\alpha,s)}{s-\beta} ds \right| + 8\sqrt{\frac{\|w\|_J}{w(0)M}} \end{aligned} \tag{3.1}$$

where $\|w\|_J$ is the essential supremum of w on J.

Proof of Theorem 3.1 assuming Lemma 3.2. Fix $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C} \setminus \mathbb{R}$. As $M \to \infty$ the righthand side of inequality (3.1) goes to zero. This is because

$$\frac{\mathbb{S}(\alpha,s)}{s-\beta} = \frac{\sin(\pi(s-\alpha))}{\pi(s-\alpha)(s-\beta)}$$

is integrable so its tail goes to zero. Since M is arbitrary, we get the limit relation in Theorem 3.1. $\square$

Proof of Lemma 3.2. By shifting the measure, we can assume that $x = 0$, which we henceforth do for ease of notation. Fix $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C} \setminus \mathbb{R}$, and fix

$M \geq 2|\beta|$. Write

$$\left|\frac{1}{\widetilde{K}_n(0,0)}\int \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)},t\right)}{t-\frac{\beta}{\widetilde{K}_n(0,0)}}d\mu(t)-\int\frac{\mathbb{S}(\alpha,s)}{s-\beta}ds\right|$$

$$\leq \left|\frac{1}{\widetilde{K}_n(0,0)}\int \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)},t\right)}{t-\frac{\beta}{\widetilde{K}_n(0,0)}}d\mu(t)-\int_{-M}^{M}\frac{\mathbb{S}(\alpha,s)}{s-\beta}ds\right|+\left|\int_{\mathbb{R}\setminus[-M,M]}\frac{\mathbb{S}(\alpha,s)}{s-\beta}ds\right|.$$

To prove the Lemma, it is enough to show that

$$\begin{aligned}&\limsup_{n\to\infty}\left|\frac{1}{\widetilde{K}_n(0,0)}\int \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)},t\right)}{t-\frac{\beta}{\widetilde{K}_n(0,0)}}d\mu(t)-\int_{-M}^{M}\frac{\mathbb{S}(\alpha,s)}{s-\beta}ds\right|\\&\leq 8\sqrt{\frac{\|w\|_J}{w(0)M}}.\end{aligned}\tag{3.2}$$

Let $I_n=[-\frac{M}{\widetilde{K}_n(0,0)},\frac{M}{\widetilde{K}_n(0,0)}]$, and write

$$\left|\frac{1}{\widetilde{K}_n(0,0)}\int \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)},t\right)}{t-\frac{\beta}{\widetilde{K}_n(0,0)}}d\mu(t)-\int_{-M}^{M}\frac{\mathbb{S}(\alpha,s)}{s-\beta}ds\right|$$

$$\leq \left|\frac{1}{\widetilde{K}_n(0,0)}\int_{I_n} \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)},t\right)}{t-\frac{\beta}{\widetilde{K}_n(0,0)}}d\mu(t)-\int_{-M}^{M}\frac{\mathbb{S}(\alpha,s)}{s-\beta}ds\right|$$

$$+\left|\frac{1}{\widetilde{K}_n(0,0)}\int_{\mathbb{R}\setminus I_n} \frac{K_n\left(\frac{a}{\widetilde{K}_n(0,0)},t\right)}{t-\frac{\beta}{\widetilde{K}_n(0,0)}}d\mu(t)\right|.$$

Consider first the first term in the righthand side of the inequality above,

$$\left|\frac{1}{\widetilde{K}_n(0,0)}\int_{I_n} \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)},t\right)}{t-\frac{\beta}{\widetilde{K}_n(0,0)}}d\mu(t)-\int_{-M}^{M}\frac{\mathbb{S}(\alpha,s)}{s-\beta}ds\right|.$$

Since 0 is not a pure point of μ, it follows that $K_n(0,0)\to\infty$. Thus, for sufficiently large n, $I_n\subseteq J$. We can therefore write

$$\left|\frac{1}{\widetilde{K}_n(0,0)}\int_{I_n} \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)},t\right)}{t-\frac{\beta}{\widetilde{K}_n(0,0)}}d\mu(t)-\int_{-M}^{M}\frac{\mathbb{S}(\alpha,s)}{s-\beta}ds\right|$$

$$=\left|\frac{1}{\widetilde{K}_n(0,0)}\int_{I_n} \frac{K_n\left(\frac{a}{\widetilde{K}_n(0,0)},t\right)w(t)dt}{t-\frac{\beta}{\widetilde{K}_n(0,0)}}-\int_{-M}^{M}\frac{\mathbb{S}(\alpha,s)}{s-\beta}ds\right|,$$

and by transformation of variables $t = \frac{s}{\widetilde{K}_n(0,0)}$ we get

$$\left| \frac{1}{\widetilde{K}_n(0,0)} \int_{I_n} \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)}, t\right) w(t)dt}{t - \frac{\beta}{\widetilde{K}_n(0,0)}} - \int_{-M}^{M} \frac{\mathbb{S}(\alpha, s)}{s - \beta} ds \right|$$
$$= \left| \int_{-M}^{M} \frac{1}{s-\beta} \left(\frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)}, \frac{s}{\widetilde{K}_n(0,0)}\right) w\left(\frac{s}{\widetilde{K}_n(0,0)}\right)}{K_n(0,0)w(0)} - \mathbb{S}(\alpha, s) \right) ds \right|.$$

The righthand side of the equality above goes to zero as $n \to \infty$. This follows since K_n has a unifrom universal limit at 0, since w is essentially bounded on J and since 0 is a Lebesgue point of w (note that $|s - \beta| \geq |\operatorname{Im}(\beta)| > 0$).

It remains to show that the inequality

$$\limsup_{n\to\infty} \left| \frac{1}{\widetilde{K}_n(0,0)} \int_{\mathbb{R}\setminus I_n} \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)}, t\right)}{t - \frac{\beta}{\widetilde{K}_n(0,0)}} d\mu(t) \right| \leq 8\sqrt{\frac{\|w\|_J}{w(0)M}}. \tag{3.3}$$

holds. First, by the Cauchy-Schwarz inequality we obtain

$$\left| \frac{1}{\widetilde{K}_n(0,0)} \int_{\mathbb{R}\setminus I_n} \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)}, t\right)}{t - \frac{\beta}{\widetilde{K}_n(0,0)}} d\mu(t) \right|$$
$$\leq \left| \frac{1}{\widetilde{K}_n(0,0)} \right| \left(\int_{\mathbb{R}\setminus I_n} \frac{d\mu(t)}{\left(t - \frac{\beta}{\widetilde{K}_n(0,0)}\right)^2} \right)^{1/2} \left(\int_{\mathbb{R}} K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)}, t\right)^2 d\mu(t) \right)^{1/2}.$$

Second, the above inequality, and the reproducing property of the kernel imply

$$\left| \frac{1}{\widetilde{K}_n(0,0)} \int_{\mathbb{R}\setminus I_n} \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)}, t\right)}{t - \frac{\beta}{\widetilde{K}_n(0,0)}} d\mu(t) \right|$$
$$\leq \left| \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)}, \frac{\alpha}{\widetilde{K}_n(0,0)}\right)^{1/2}}{w(0)K_n(0,0)} \right| \left(\int_{\mathbb{R}\setminus I_n} \frac{d\mu(t)}{\left(t - \frac{\beta}{\widetilde{K}_n(0,0)}\right)^2} \right)^{1/2}.$$

Note that for sufficiently large n,

$$\left| \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)}, \frac{\alpha}{\widetilde{K}_n(0,0)}\right)^{1/2}}{w(0)K_n(0,0)} \right| \leq \frac{2}{w(0)K_n(0,0)^{1/2}},$$

by assumption (iii) in the statement of Theorem 1.2. Thus we get, for sufficiently large n,

$$\begin{aligned}&\left|\frac{1}{\widetilde{K}_n(0,0)}\int_{\mathbb{R}\setminus I_n}\frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)},t\right)}{t-\frac{\beta}{\widetilde{K}_n(0,0)}}d\mu(t)\right| \\ &\leq \frac{2}{w(0)K_n(0,0)^{1/2}}\left(\int_{\mathbb{R}\setminus I_n}\frac{d\mu(t)}{\left(t-\frac{\beta}{\widetilde{K}_n(0,0)}\right)^2}\right)^{1/2}.\end{aligned} \tag{3.4}$$

We are left with estimating

$$\int_{\mathbb{R}\setminus I_n}\frac{d\mu(t)}{\left(t-\frac{\beta}{\widetilde{K}_n(0,0)}\right)^2}.$$

First note that, since $M \geq 2|\beta|$, we get for every t with $|t| \geq \frac{M}{\widetilde{K}_n(0,0)}$,

$$|t-\frac{\beta}{\widetilde{K}_n(0,0)}| \geq \frac{t}{2}.$$

Therefore,

$$\int_{\mathbb{R}\setminus I_n}\frac{d\mu(t)}{\left(t-\frac{\beta}{\widetilde{K}_n(0,0)}\right)^2} \leq \int_{\mathbb{R}\setminus I_n}\frac{4d\mu(t)}{t^2}.$$

Now we split the integral in the righthand side of the inequality above. Let $H_n = [-\frac{M}{\widetilde{K}_n(0,0)^{1/3}}, \frac{M}{\widetilde{K}_n(0,0)^{1/3}}]$, and again note that for sufficiently large n, $H_n \subseteq J$. Clearly, $I_n \subseteq H_n$, so we can write

$$\mathbb{R}\setminus I_n = (\mathbb{R}\setminus H_n) \cup (H_n \setminus I_n)\,.$$

We split the integral accordingly,

$$\begin{aligned}\left|\int_{\mathbb{R}\setminus I_n}\frac{4d\mu(t)}{t^2}\right| &= \left|\int_{\mathbb{R}\setminus H_n}\frac{4d\mu(t)}{t^2}+\int_{H_n\setminus I_n}\frac{4d\mu(t)}{t^2}\right| \\ &= \left|\int_{\mathbb{R}\setminus H_n}\frac{4d\mu(t)}{t^2}+\int_{H_n\setminus I_n}\frac{4w(t)dt}{t^2}\right| \\ &\leq \left|\int_{\mathbb{R}\setminus H_n}\frac{4d\mu(t)}{t^2}\right| + 4\|w\|_J\cdot\left|\int_{H_n\setminus I_n}\frac{dt}{t^2}\right|,\end{aligned}$$

where we have used the fact that $H_n \subseteq J$ to write $d\mu(t) = w(t)dt$ there. The proof of Lemma 3.2 is almost complete. For the first integral on the left, note that for $t \notin H_n$, $t^2 \geq \frac{M^2}{\widetilde{K}_n(0,0)^{2/3}}$. Therefore, we have (taking into account that

μ is a probability measure)

$$\left|\int_{\mathbb{R}\setminus H_n} \frac{4d\mu(t)}{t^2}\right| \leq 4\frac{\widetilde{K}_n(0,0)^{2/3}}{M^2}\int d\mu(t) = 4\frac{\widetilde{K}_n(0,0)^{2/3}}{M^2}.$$

For the second integral, integration of $\frac{1}{t^2}$ gives (recall the definition of I_n)

$$4\|w\|_J \cdot \left|\int_{H_n\setminus I_n} \frac{dt}{t^2}\right| \leq 4\|w\|_J \cdot \int_{\mathbb{R}\setminus I_n} \frac{dt}{t^2} = 8\|w\|_J \frac{\widetilde{K}_n(0,0)}{M}.$$

Plugging these estimates into (3.4), we see

$$\begin{aligned}
&\left|\frac{1}{\widetilde{K}_n(0,0)}\int_{\mathbb{R}\setminus I_n} \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)}, t\right)}{t-\frac{\beta}{\widetilde{K}_n(0,0)}} d\mu(t)\right| \\
&\leq \frac{8}{w(0)K_n(0,0)^{1/2}}\left(\frac{\widetilde{K}_n(0,0)^{2/3}}{M^2} + \|w\|_J \frac{\widetilde{K}_n(0,0)}{M}\right)^{1/2} \qquad (3.5)\\
&= \frac{8}{\sqrt{w(0)}}\left(\frac{1}{M^2\widetilde{K}_n(0,0)^{1/3}} + \|w\|_J \frac{1}{M}\right)^{1/2}
\end{aligned}$$

which immediately shows that

$$\limsup_{n\to\infty}\left|\frac{1}{\widetilde{K}_n(0,0)}\int_{\mathbb{R}\setminus I_n} \frac{K_n\left(\frac{\alpha}{\widetilde{K}_n(0,0)}, t\right)}{t-\frac{\beta}{\widetilde{K}_n(0,0)}} d\mu(t)\right| \leq 8\sqrt{\frac{\|w\|_J}{w(0)M}}.$$

□

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